On multilattice counterparts of **MNT4**, **S4**, and **S5**

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Abstract

In this report, we are going to introduce three recently developed modal multilattice logics based on **MNT4**, **S4**, and **S5** in the form of cut-free sequent and hypersequent calculi as well as in the form of algebraic semantics.

Multilattice logic $\mathbf{ML_n}$ was designed by Shramko [9] in order to generalize frameworks of Arieli and Avron's bilattice logic [1], Shramko and Wansing's trilattice logic [10], and Zaitsev's tetralattice logic [11]. Modal multilattice logic $\mathbf{MML_n}$ was developed by Kamide and Shramko [7]. They expected that this logic will be a multilattice version of **S4**. However, as argued in [5, 4], it is not really the case. **S4** proves the interdefinability of necessity and possibility modal operators, while, as follows from the embedding theorem of **S4** into $\mathbf{MML_n}$ [7], the latter logic does not have the interdefinability axioms. Moreover, the algebraic structure suggested by Kamide and Shramko is too weak to be an adequaete semantics for $\mathbf{MML_n}$ (see [5] for the details). The closure and interior operators introduced by Kamide and Shramko are rather multilattice versions of Tarski's operators (which are suitable for **MNT4**), than Kuratowski ones (which are needed for **S4**). It has motivated us to present a genuine multilattice version of **S4** based on Kuratowski's closure and interior operators (we call this logic $\mathbf{MML_n^{S4}}$) and a multilattice version of **MNT4** based on Tarski's operator (we call this logic $\mathbf{MML_n^{MNT4}}$).

Moreover, we consider one more logic: $\mathbf{MML_n^{S5}}$ which is a multilattice version of **S5**. Its algebraic semantics is based on Halmos closure and interior operators. What is important in the case of **S5** (since we a interested not only in algebraic, but proof-theoretical aspects of multilattice logics), **S5** has an impressive amount of various proof systems. In particular, it has at least eight various cut-free hypersequent calculi (see [6] for the latest one and [2] for a survey of the others). This feature of **S5** makes it a good candidate for the development on its base of non-standard modal logics (for example, multilattice modal logics).

Let us introduce the notion of multilattice.

Definition 1. [7, p. 319, Definitions 2.1 and 2.2] A *multilattice* is a structure $\mathcal{M}_n = \langle S, \leqslant_1, \ldots, \leqslant_n\rangle$, where $n > 1$, $S \neq \emptyset$, $\leqslant_1, \ldots, \leqslant_n$ are partial orders such that $\langle S, \leqslant_1\rangle, \ldots, \langle S, \leqslant_n\rangle$ are lattices with the corresponding pairs of meet and join operators $\langle \cap_1, \cup_1\rangle, \ldots, \langle \cap_n, \cup_n\rangle$ as well as the corresponding j-inversion operators $-_1, \ldots, -_n$ which satisfy the following conditions, for each $j, k \leqslant n$, $j \neq k$, and $a, b \in S$:

$a \leqslant_j b$ implies $-_j b \leqslant_j -_j a$; $\quad$ $a \leqslant_k b$ implies $-_j a \leqslant_k -_j b$; $\quad$ $-_j -_j a = a$.

Definition 2 (Ultralogical multilattice)**.** [7, p. 319, Definitions 2.3 and 2.4] A pair $\langle \mathcal{M}_n, \mathcal{U}_n\rangle$ is called an *ultralogical multilattice* iff $\mathcal{M}_n = \langle S, \leqslant_1, \ldots, \leqslant_n\rangle$ is a multilattice and $\mathcal{U}_n \subsetneq S$ satisfies the following conditions, for each $j, k \leqslant n$, $j \neq k$, and $a, b \in S$:

- $a \cap_j b \in \mathcal{U}_n$ iff $a \in \mathcal{U}_n$ and $b \in \mathcal{U}_n$ ($\mathcal{U}_n$ is a multifilter on $\mathcal{M}_n$);
- $a \cup_j b \in \mathcal{U}_n$ iff $a \in \mathcal{U}_n$ or $b \in \mathcal{U}_n$ ($\mathcal{U}_n$ is a prime multifilter on $\mathcal{M}_n$);
- $a \in \mathcal{U}_n$ iff $-_j -_k a \notin \mathcal{U}_n$ ($\mathcal{U}_n$ is an ultramultifilter on $\mathcal{M}_n$).

The formulas of $\mathbf{ML_n}$ are built from the set $\mathcal{P} = \{p_n \mid n \in \mathbb{N}\}$ of propositional variables, negations $\neg_1, \ldots, \neg_n$, conjunctions $\wedge_1, \ldots, \wedge_n$, and disjunctions $\vee_1, \ldots, \vee_n$. A valuation v is defined as a mapping from $\mathcal{P}$ to S. It is extended into complex formulas as follows: $v(\neg_j \phi) = -_j v(\phi)$, $v(\phi \wedge_j \psi) = v(\phi) \cap_j v(\psi)$, and $v(\phi \vee_j \psi) = v(\phi) \cup_j v(\psi)$. The entailment relation is defined as follows:

$\Gamma \models_{\mathbf{ML}_n} \Delta$ iff for each De Morgan ultralogical multilattice $\langle \mathcal{M}_n, \mathcal{U}_n \rangle$ and each valuation v, it holds that if $v(\gamma) \in \mathcal{U}_n$ (for each $\gamma \in \Gamma$), then $v(\delta) \in \mathcal{U}_n$ (for some $\delta \in \Delta$).

In the next definition we adopt the notions of Tarski, Kuratowski, and Halmos closure and interior operators for the multilattice case (we follow Cattaneo and Ciucci [3]).

Definition 3. We say that a multilattice $\mathcal{M}_n = \langle S, \leqslant_1, \ldots, \leqslant_n \rangle$ have *Tarski* operators iff for each $j \leqslant n$ the unary operators of *interior* I_j and *closure* C_j can be defined on S and satisfy the subsequent conditions ($a, b, c \in S$, $1 := c \cup_j \neg_j \neg_k c$, $0 := c \cap_j \neg_j \neg_k c$, $k \neq j$):

$I_j(a) \leqslant_j a$;	$C_j(a) \cup_j C_j(b) \leqslant_j C_j(a \cup_j b)$;	$-_k I_j(a) = I_j(-_k a)$;
$I_j(a) = I_j I_j(a)$;	$I_j(1) = 1$;	$-_k C_j(a) = C_j(-_k a)$;
$I_j(a \cap_j b) \leqslant_j I_j(a) \cap_j I_j(b)$;	$C_j(0) = 0$;	$I_j(a) = -_j -_k C_j(-_j -_k a)$;
$a \leqslant_j C_j(a)$;	$-_j I_j(a) = C_j(-_j a)$;	$C_j(a) = -_j -_k I_j(-_j -_k a)$.
$C_j(a) = C_j C_j(a)$;	$-_j C_j(a) = I_j(-_j a)$;	

Tarski operators are said to be *Kuratowski* ones iff the subsequent conditions are fulfilled: $I_j(a \cap_j b) = I_j(a) \cap_j I_j(b)$ and $C_j(a) \cup_j C_j(b) = C_j(a \cup_j b)$. Kuratowski operators are said to be *Halmos* ones iff the subsequent conditions are fulfilled: $I_j(-_j I_j(a)) = -_j I_j(a)$ and $C_j(-_j C_j(a)) = -_j C_j(a)$.

The formulas of modal multilattice logics are built not only from propositional connectives, but necessity operators $\Box_1, \ldots, \Box_n$ and possibility operators $\Diamond_1, \ldots, \Diamond_n$. The entailment relation in modal multilattice logics is understood in the following way:

$\Gamma \models \Delta$ in $\mathbf{MML}_n^{\mathbf{MNT4}}$ (resp., $\mathbf{MML}_n^{\mathbf{S4}}$, $\mathbf{MML}_n^{\mathbf{S5}}$) iff for each ultralogical multilattice $\langle \mathcal{M}_n, \mathcal{U}_n \rangle$ with Tarski (resp., Kuratowski, Halmos) operators and each valuation v, it holds that if it holds that if $v(\gamma) \in \mathcal{U}_n$ (for each $\gamma \in \Gamma$), then $v(\delta) \in \mathcal{U}_n$ (for some $\delta \in \Delta$).

Let us introduce a sequent calculus for the logic $\mathbf{MML_n^{MNT4}}$. By a sequent we understood a pair written as $\Gamma \Rightarrow \Delta$, where Γ, Δ are finite sets of formulas. In what follows, the letter π denotes a set which is either empty or consists of exactly one formula from the list $\Box_j \psi, \neg_j \Diamond_j \psi, \neg_k \Box_j \psi$, where $k \neq j$; the letter δ denotes a set which is either empty or consists of exactly one formula from the list $\Diamond_j \psi, \neg_j \Box_j \psi, \neg_k \Diamond_j \psi$, where $k \neq j$. The axioms are as follows:

$$\text{(A) } \phi \Rightarrow \phi \qquad (\text{A}_\neg)\ \neg_j \phi \Rightarrow \neg_j \phi$$

The structural rules are cut (which is admissible) and weakening. The non-negated logical rules are as follows:

$$(\wedge_j \Rightarrow)\ \frac{\phi, \psi, \Gamma \Rightarrow \Delta}{\phi \wedge_j \psi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \wedge_j)\ \frac{\Gamma \Rightarrow \Delta, \phi \qquad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \phi \wedge_j \psi}$$

$$(\vee_j \Rightarrow)\ \frac{\phi, \Gamma \Rightarrow \Delta \qquad \psi, \Gamma \Rightarrow \Delta}{\phi \vee_j \psi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \vee_j)\ \frac{\Gamma \Rightarrow \Delta, \phi, \psi}{\Gamma \Rightarrow \Delta, \phi \vee_j \psi}$$

The jj-negated logical rules are as follows:

$$(\neg_j\wedge_j\Rightarrow)\ \frac{\neg_j\phi,\Gamma\Rightarrow\Delta \qquad \neg_j\psi,\Gamma\Rightarrow\Delta}{\neg_j(\phi\wedge_j\psi),\Gamma\Rightarrow\Delta} \qquad (\Rightarrow\neg_j\wedge_j)\ \frac{\Gamma\Rightarrow\Delta,\neg_j\phi,\neg_j\psi}{\Gamma\Rightarrow\Delta,\neg_j(\phi\wedge_j\psi)}$$

$$(\neg_j\vee_j\Rightarrow)\ \frac{\neg_j\phi,\neg_j\psi,\Gamma\Rightarrow\Delta}{\neg_j(\phi\vee_j\psi),\Gamma\Rightarrow\Delta} \qquad (\Rightarrow\neg_j\vee_j)\ \frac{\Gamma\Rightarrow\Delta,\neg_j\phi \qquad \Gamma\Rightarrow\Delta,\neg_j\psi}{\Gamma\Rightarrow\Delta,\neg_j(\phi\vee_j\psi)}$$

$$(\neg_j\neg_j\Rightarrow)\ \frac{\phi,\Gamma\Rightarrow\Delta}{\neg_j\neg_j\phi,\Gamma\Rightarrow\Delta} \qquad (\Rightarrow\neg_j\neg_j)\ \frac{\Gamma\Rightarrow\Delta,\phi}{\Gamma\Rightarrow\Delta,\neg_j\neg_j\phi}$$

The kj-negated logical rules as follows:

$$(\neg_k\wedge_j\Rightarrow)\ \frac{\neg_k\phi,\neg_k\psi,\Gamma\Rightarrow\Delta}{\neg_k(\phi\wedge_j\psi),\Gamma\Rightarrow\Delta} \qquad (\Rightarrow\neg_k\wedge_j)\ \frac{\Gamma\Rightarrow\Delta,\neg_k\phi \qquad \Gamma\Rightarrow\Delta,\neg_k\psi}{\Gamma\Rightarrow\Delta,\neg_k(\phi\wedge_j\psi)}$$

$$(\neg_k\vee_j\Rightarrow)\ \frac{\neg_k\phi,\Gamma\Rightarrow\Delta \qquad \neg_k\psi,\Gamma\Rightarrow\Delta}{\neg_k(\phi\vee_j\psi),\Gamma\Rightarrow\Delta} \qquad (\Rightarrow\neg_k\vee_j)\ \frac{\Gamma\Rightarrow\Delta,\neg_k\phi,\neg_k\psi}{\Gamma\Rightarrow\Delta,\neg_k(\phi\vee_j\psi)}$$

$$(\neg_k\neg_j\Rightarrow)\ \frac{\Gamma\Rightarrow\Delta,\phi}{\neg_k\neg_j\phi,\Gamma\Rightarrow\Delta} \qquad (\Rightarrow\neg_k\neg_j)\ \frac{\phi,\Gamma\Rightarrow\Delta}{\Gamma\Rightarrow\Delta,\neg_k\neg_j\phi}$$

The non-negated modal rules are as follows:

$$(\Box_j\Rightarrow)\ \frac{\phi,\Gamma\Rightarrow\Delta}{\Box_j\phi,\Gamma\Rightarrow\Delta} \quad (\Rightarrow\Diamond_j)\ \frac{\Gamma\Rightarrow\Delta,\phi}{\Gamma\Rightarrow\Delta,\Diamond_j\phi} \quad (\Rightarrow\Box_j)\ \frac{\pi\Rightarrow\Diamond_j\Lambda,\phi}{\pi\Rightarrow\Diamond_j\Lambda,\Box_j\phi} \quad (\Diamond_j\Rightarrow)\ \frac{\phi,\Box_j\Lambda\Rightarrow\delta}{\Diamond_j\phi,\Box_j\Lambda\Rightarrow\delta}$$

The jj-negated modal logical rules:

$$(\Rightarrow\neg_j\Box_j)\ \frac{\Gamma\Rightarrow\Delta,\neg_j\phi}{\Gamma\Rightarrow\Delta,\neg_j\Box_j\phi} \qquad (\neg_j\Diamond_j\Rightarrow)\ \frac{\neg_j\phi,\Gamma\Rightarrow\Delta}{\neg_j\Diamond_j\phi,\Gamma\Rightarrow\Delta}$$

$$(\neg_j\Box_j\Rightarrow)\ \frac{\neg_j\phi,\Box_j\Lambda\Rightarrow\delta}{\neg_j\Box_j\phi,\Box_j\Lambda\Rightarrow\delta} \qquad (\Rightarrow\neg_j\Diamond_j)\ \frac{\pi\Rightarrow\Diamond_j\Lambda,\neg_j\phi}{\pi\Rightarrow\Diamond_j\Lambda,\neg_j\Diamond_j\phi}$$

The kj-negated modal logical rules:

$$(\neg_k\Box_j\Rightarrow)\ \frac{\neg_k\phi,\Gamma\Rightarrow\Delta}{\neg_k\Box_j\phi,\Gamma\Rightarrow\Delta} \qquad (\Rightarrow\neg_k\Diamond_j)\ \frac{\Gamma\Rightarrow\Delta,\neg_k\phi}{\Gamma\Rightarrow\Delta,\neg_k\Diamond_j\phi}$$

$$(\Rightarrow\neg_k\Box_j)\ \frac{\pi\Rightarrow\Diamond_j\Lambda,\neg_k\phi}{\pi\Rightarrow\Diamond_j\Lambda,\neg_k\Box_j\phi} \qquad (\neg_k\Diamond_j\Rightarrow)\ \frac{\neg_k\phi,\Box_j\Lambda\Rightarrow\delta}{\neg_k\Diamond_j\phi,\Box_j\Lambda\Rightarrow\delta}$$

A sequent calculus for $\mathbf{MML}_n^{\mathbf{S4}}$ [4] is obtained from the one for $\mathbf{MML}_n^{\mathbf{MNT4}}$ by the replacement in each of modal rule the letters δ and π, respectively, with the sets $\{\Box_j\Gamma_1,\neg_j\Diamond_j\Gamma_2,\neg_k\Box_j\Gamma_3\}$ and $\{\Diamond_j\Delta_1,\neg_j\Box_j\Delta_2,\neg_k\Diamond_j\Delta_3\}$ (where $k\neq j$) as well as $\Box_j\Lambda$ and $\Diamond_j\Lambda$, respectively, with $\{\Box_j\Lambda_1,\neg_j\Diamond_j\Lambda_2,\neg_k\Box_j\Lambda_3\}$ and $\{\Diamond_j\Lambda_1,\neg_j\Box_j\Lambda_2,\neg_k\Diamond_j\Lambda_3\}$. Because of the lack of space, we are not able to present here a hypersequent calculus for $\mathbf{MML}_n^{\mathbf{S5}}$ based on Restall's hypersequent calculus for **S5** [8], but the reader may find it in [4]. All the calculi for modal multilattice logics are show to be sound, complete, and cut-free.

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