Infinitary Action Logic with Exponentiation

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The Lambek calculus [9] was introduced as a logical framework for describing natural language syntax. In order to be useful for such applications, the Lambek calculus is highly substructural, including neither contraction, nor weakening, nor permutation structural rules. The only structural rule kept is implicit associativity. From a modern point of view [1], the Lambek calculus can be considered as a non-commutative intuitionstic version of Girard's linear logic [3]. Thus, the Lambek can be further extended by linear logic connectives, such as additives and (sub)exponentials.

The derivability problem for the basic Lambek calculus is NP-complete [13]. The multiplicative-additive Lambek calculus (viz., the Lambek calculus extended with additive conjunction and disjunction, denoted by MALC) is PSPACE-hard [4, 6]. Extending the Lambek calculus with an exponential modality yields an undecidable (Σ_1^0-complete) system [10]. A more fine-grained system can be obtained by extending MALC with a family of structural modalities, called subexponentials, cf. [11] Such a non-commutative version of the subexponential extension of linear logic was studied by Kanovich et al. [5]. The Lambek calculus with subexponentials is also undecidable, provided that at least one of the subexponentials allows the rule of non-local contraction.

Action logic, or the Lambek calculus with additives further extended with iteration (Kleene star), originates in the works of Pratt [14] and Kozen [7]. Buszkowski and Palka [2, 12] considered a stronger version of action logic, where iteration is governed by an ω-rule instead of inductive-style axioms. This system is called *infinitary action logic.* Buszkowski and Palka proved that it is Π_1^0-complete (thus, in particular, not computably enumerable).

We study an extension of MALC with *both* Kleene star and a family subexponentials. This extension is called *infinitary action logic with exponentiation* and denoted by $!\mathrm{ACT}_\omega$.

Formulae of $!\mathrm{ACT}_\omega$ are built from propositional *variables* ($\mathrm{Var} = \{p_1, p_2, p_3, \ldots\}$) and the *multiplicative unit (truth)* constant $\mathbf{1}$ using the following binary connectives:

- multiplicative connectives: *left implication* $\multimap$, *right implication* $\multimapinv$, and *product (multiplicative conjunction)* $\otimes$;
- additive connectives: *conjunction* $\&$ and *disjunction* $\oplus$

and the following unary connectives:

- *iteration (Kleene star)* *;
- *subexponentials:* we fix a partially ordered set $\langle \mathcal{I}, \preceq \rangle$ of subexponential labels, and three subsets of $\mathcal{I}$, called $\mathcal{W}$, $\mathcal{C}$, and $\mathcal{E}$, upwardly closed w.r.t. $\preceq$

  For each $s \in \mathcal{I}$ we introduce a unary connective $!^s$.

  Intuitively, $\mathcal{W}$, $\mathcal{C}$, and $\mathcal{E}$ mean the sets of subexponentials for which we allow weakening, contraction, and permutation (exchange) rules respectively.

The axioms and rules of $!\mathrm{ACT}_\omega$ are as follows:

$$\frac{}{A \vdash A}\ (\mathrm{id})$$

$$\frac{\Pi \to A \quad \Gamma, B, \Delta \vdash C}{\Gamma, \Pi, A \multimap B, \Delta \vdash C}\ (\multimap\vdash) \qquad \frac{A, \Pi \vdash B}{\Pi \vdash A \multimap B}\ (\vdash\multimap)$$

$$\frac{\Pi \vdash A \quad \Gamma, B, \Delta \vdash C}{\Gamma, B \circ\!\!- A, \Pi, \Delta \vdash C}\ (\circ\!\!-\vdash) \qquad \frac{\Pi, A \vdash B}{\Pi \vdash B \circ\!\!- A}\ (\vdash\circ\!\!-)$$

$$\frac{\Gamma, A, B, \Delta \to C}{\Gamma, A \otimes B, \Delta \vdash C}\ (\otimes\vdash) \qquad \frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}\ (\vdash\otimes)$$

$$\frac{\Gamma, \Delta \to C}{\Gamma, \mathbf{1}, \Delta \to C}\ (\mathbf{1}\vdash) \qquad \frac{}{\vdash \mathbf{1}}\ (\vdash\mathbf{1})$$

$$\frac{\Gamma, A_1, \Delta \vdash C \quad \Gamma, A_2, \Delta \vdash C}{\Gamma, A_1 \oplus A_2, \Delta \vdash C}\ (\oplus\vdash) \qquad \frac{\Pi \to A_i}{\Pi \to A_1 \oplus A_2}\ (\vdash\oplus)_i,\ i = 1, 2$$

$$\frac{\Gamma, A_i, \Delta \to C}{\Gamma, A_1 \,\&\, A_2, \Delta \to C}\ (\&\vdash)_i,\ i = 1, 2 \qquad \frac{\Pi \to A_1 \quad \Pi \to A_2}{\Pi \to A_1 \,\&\, A_2}\ (\vdash\&)$$

$$\frac{(\Gamma, A^n, \Delta \vdash C)_{n\in\mathbb{N}}}{\Gamma, A^*, \Delta \vdash C}\ ({}^*\vdash)_\omega \qquad \frac{\Pi_1 \to A \quad \dots \quad \Pi_n \vdash A}{\Pi_1, \dots, \Pi_n \vdash A^*}\ (\vdash{}^*)_n,\ n \geqslant 0$$

$$\frac{\Gamma, A, \Delta \vdash C}{\Gamma, !^s A, \Delta \vdash C}\ (!\vdash) \qquad \frac{!^{s_1} A_1, \dots, !^{s_n} A_n \vdash B}{!^{s_1} A_1, \dots, !^{s_n} A_n \vdash !^s B}\ (\vdash !),\ s_i \succeq s$$

$$\frac{\Gamma, A, \Delta \to C}{\Gamma, !^w A, \Delta \to C}\ (\mathrm{weak}),\ w \in \mathcal{W}$$

$$\frac{\Gamma, \Phi, !^e A, \Delta \vdash C}{\Gamma, !^e A, \Phi, \Delta \vdash C}\ (\mathrm{perm})_1,\ e \in \mathcal{E} \qquad \frac{\Gamma, !^e A, \Phi, \Delta \vdash C}{\Gamma, \Phi, !^e A, \Delta \vdash C}\ (\mathrm{perm})_2,\ e \in \mathcal{E}$$

$$\frac{\Gamma, !^c A, \Phi, !^c A, \Delta \vdash C}{\Gamma, !^c A, \Phi, \Delta \vdash C}\ (\mathrm{ncontr})_1,\ c \in \mathcal{C} \qquad \frac{\Gamma, !^c A, \Phi, !^c A, \Delta \vdash C}{\Gamma, \Phi, !^c A, \Delta \vdash C}\ (\mathrm{ncontr})_2,\ c \in \mathcal{C}$$

$$\frac{\Pi \vdash A \quad \Gamma, A, \Delta \vdash C}{\Gamma, \Pi, \Delta \vdash C}\ (\mathrm{cut})$$

Since (ncontr) and (weak) derive (perm), we explicitly postulate $\mathcal{W} \cap \mathcal{C} \subseteq \mathcal{E}$.

Derivations in $!\mathrm{ACT}_\omega$ are trees which can be infinitely branching, but should be well-founded (that is, infinite paths are not allowed).

The cut rule is eliminable, which is established by a juxtaposition of two arguments. The first one is cut elimination in infinitary action logic, performed by Palka [12] using transfinite induction. The second one is cut elimination is the subexponential extension of MALC by Kanovich et al. [5], using a version of Gentzen's mix rule.

Our main result is that a combination of exponential and Kleene star yields a system of hyperarithmetical complexity:

Theorem 1. *If $\mathcal{C} \neq \varnothing$, then the derivability problem in* $!\mathrm{ACT}_\omega$ *is* Π^1_1*-complete.*

The proof of the lower bound, Π^1_1-hardness, is based on encoding Kozen's result on the complexity of Horn theories for *-continuous Kleene algebras [8]. The upper bound is established by quite a general argument, based on the form of the rules and derivations in the calculus.

Another measure of complexity of $!\mathrm{ACT}_\omega$ is its *closure ordinal.* The closure ordinal is defined as follows. Let $\mathscr{D}$ be the *immediate derivability operator.* The $\mathscr{D}$ operator is a mapping of sets of sequents into sets of sequents. For a set of sequents S and a sequent s we have $s \in \mathscr{D}(S)$ if and only if either $s \in S$, or s is an axiom, or s is obtained by one of the inference rules from sequents belonging to S.

By $\mathscr{D}^\alpha$, for an ordinal α, we denote the α-th transfinite iteration of $\mathscr{D}$. The closure ordinal is the smallest ordinal α such that $\mathscr{D}^\alpha(\varnothing) = \mathscr{D}^{\alpha+1}(\varnothing)$. The existence of such α follows from the Knaster–Tarski theorem.

We compute the closure ordinal for $!\mathrm{ACT}_\omega$:

Theorem 2. *If $\mathcal{C} \neq \varnothing$, the closure ordinal for* $!\mathrm{ACT}_\omega$ *(for the $\mathscr{D}$ operator defined above using axioms and rules of* $!\mathrm{ACT}_\omega$*) is* ω_1^{CK}*, that is, the smallest non-computable ordinal, known as the Church–Kleene ordinal.*

Thus, we have established exact complexity bounds for $!\mathrm{ACT}_\omega$, both in terms of the complexity class for the derivability problem and in terms of the closure ordinal of the immediate derivability operator. Complexity of naturally arising fragments of $!\mathrm{ACT}_\omega$, with $\mathcal{C} = \varnothing$ (that is, where no subexponential allows contraction) or where $!^c$, $c \in \mathcal{C}$, cannot be applied to formulae containing the Kleene star, is left for future research.

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