On Globally Sound Analytic Calculi for Quantifier Macros

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Abstract

In this work we present a methodology to construct globally sound but possibly locally unsound analytic calculi for partial theories of Henkin quantifiers. It is demonstrated that locally sound analytic calculi do not exist for any reasonable fragment of the full theory of Henkin quantifiers.

Henkin introduced the general idea of dependent quantifiers extending classical first-order logic [4], cf. [5] for an overview. This leads to the notion of a partially ordered quantifier with m universal quantifiers and n existential quantifiers, where F is a function that determines for each existential quantifier on which universal quantifiers it depends (m and n may be any finite number). The simplest Henkin quantifier that is not definable in ordinary first-order logic is the quantifier Q_H binding four variables in a formula. A formula A using Q_H can be written as $A_H = \begin{pmatrix} \forall x & \exists u \\ \forall y & \exists v \end{pmatrix} A(x, y, u, v)$. This is to be read "For every x there is a u and for every y there is a v (depending only on y)" s.t. $A(x, y, u, v)$. If the semantical meaning of this formula is given in second-order notation, the above formula is semantically equivalent to the second-order formula $\exists f \exists g \forall x \forall y A(x, y, f(x), g(y))$, where f and g are function variables. Systems of partially ordered quantification are intermediate in strength between first-order logic and second-order logic. Similar to second-order logic, first-order logic extended by Q_H is incomplete [7]. In proof theory incomplete logics are represented by partial proof systems, c.f. the wealth of approaches dealing with partial proof systems for second-order logic. However, in contrast to second-order logic only a few results deal with the proof theoretic aspect of the use of branching quantifiers in partial systems.[1]

The first step in this work is to establish an analytic function calculus with a suitable partial Henkin semantics. We choose a multiplicative function calculus based on pairs of multisets as sequents corresponding to term models and refer to this calculus as **LF**. Besides the usual propositional inference rules of **LK** the quantifier inference rules of **LF** are

- $\forall$-introduction for second-order function variables

$$\frac{A(t(t_1^*, \ldots, t_n^*))\Gamma \to \Delta}{\forall f^* A(f^*(t_1^*, \ldots, t_n^*)), \Gamma \to \Delta}\ \forall_l^n$$

  t is a term and $t_1^*, \ldots, t_n^*$ are semi-terms.

$$\frac{\Gamma \to \Delta, A(f(t_1^*, \ldots, t_n^*))}{\Gamma \to \Delta, \forall f^* A(f^*(t_1^*, \ldots, t_n^*))}\ \forall_r^n$$

[1]The most relevant paper is the work of Lopez-Escobar [6], describing a natural deduction system for Q_H. The setting is of course intuitionistic logic. The formulation of the introduction rule for Q_H corresponds to the introduction rule right in the sequent calculus developed in this paper. The system lacks an elimination rule.

f is a free function variable (eigenvariable) of arity n which does not occur in the lower sequent and $t_1^*, \ldots, t_n^*$ are semi-terms.

- $\exists$-introduction for second-order function variables

$$\frac{A(f(t_1^*, \ldots, t_n^*)), \Gamma \to \Delta}{\exists f^* A(f^*(t_1^*, \ldots, t_n^*)), \Gamma \to \Delta} \exists_l^n$$

f is a free function variable (eigenvariable) of arity n which does not occur in the lower sequent and $t_1^*, \ldots, t_n^*$ are semi-terms.

$$\frac{\Gamma \to \Delta, A(t(t_1^*, \ldots, t_n^*))}{\Gamma \to \Delta, \exists f^* A(f^*(t_1^*, \ldots, t_n^*))} \exists_r^n$$

t is a term and $t_1^*, \ldots, t_n^*$ are semi-terms.

LF is obviously cut-free complete w.r.t. term models by the usual Schütte argument and admits effective cut-elimination. The question arises why not to be content with the second-order representation of Henkin quantifiers. The answer is twofold: First of all, a lot of information can be extracted from cut-free proofs but only on first-order level. This includes (i) suitable variants of Herbrand's theorem with or without Skolemization, (ii) the construction of term-minimal cut-free proofs and (iii) the development of suitable tableaux provers. (i) fails due to the failure of second-order Skolemization, (ii) and (iii) fail because of the undecidability of second-order unification and the impossibility to obtain most general solutions.

Therefore, we construct the analytic calculus **LH** by deriving first-order rules from second-order rule macros. The language $\mathcal{L}_H$ of **LH** is based on the usual language of first-order logic with exception that the quantifiers are replaced by the quantifier Q_H. With exception of the quantifier-rules, **LH** corresponds to the calculus **LK** in a multiplicative setting. The idea is to abstract the eigenvariable conditions from the premises of the inference macros in **LF**. To obtain **LH**, we replace the quantifier rules of **LK** by

$$\frac{\Gamma \to \Delta, A(a, b, t_1, t_2)}{\Gamma \to \Delta, \begin{pmatrix} \forall x & \exists u \\ \forall y & \exists v \end{pmatrix} A(x, y, u, v)} Q_{H_r}$$

a and b are eigenvariables ($a \neq b$) not allowed to occur in the lower sequent and t_1 and t_2 are terms s.t. t_1 must not contain b and t_2 must not contain a.[2]

$$\frac{A(t_1', t_2', a, b), \Pi \to \Gamma}{\begin{pmatrix} \forall x & \exists u \\ \forall y & \exists v \end{pmatrix} A(x, y, u, v), \Pi \to \Gamma} Q_{H_{l_1}}$$

where a and b are eigenvariables ($a \neq b$) not allowed to occur in the lower sequent and t_1', t_2' are terms s.t. b does not occur in t_2' and a and b do not occur in t_1'.

$$\frac{A(t_1', t_2', a, b), \Pi \to \Gamma}{\begin{pmatrix} \forall x & \exists u \\ \forall y & \exists v \end{pmatrix} A(x, y, u, v), \Pi \to \Gamma} Q_{H_{l_2}}$$

[2]Note that such a rule was already used by Lopez-Escobar in [6].

where a and b are eigenvariables ($a \neq b$) not allowed to occur in the lower sequent and t_1', t_2' are terms s.t. a does not occur in t_1' and a and b do not occur in t_2'. [3]

Cuts in **LH** can be eliminated following Gentzen's procedure and we obtain a midsequent theorem. However, **LH** is incomplete: Assume towards a contradiction the sequent $\begin{pmatrix} \forall x & \exists u \\ \forall y & \exists v \end{pmatrix} A(x,y,u,v) \rightarrow \begin{pmatrix} \forall x & \exists u \\ \forall y & \exists v \end{pmatrix} (A(x,y,u,v) \vee C)$ is provable. Then it is provable without cuts. A cut-free derivation after deletion of weakenings and contractions has the form:

$$\cfrac{\cfrac{A(a,b,c,d) \rightarrow A(a,b,c,d)}{A(a,b,c,d) \rightarrow A(a,b,c,d) \vee C}}{\vdots}$$

Due to the mixture of strong and weak positions in Q_H none of Q_{H_r}, $Q_{H_{l_1}}$, $Q_{H_{l_2}}$ can be applied.

The inherent incompleteness of **LH** even for trivial statements is a consequence of the fact that Q_H represents a quantifier inference macro combining quantifiers in a strong and a weak position. This phenomenon occurs already on the level of usual first-order logic when quantifiers defined by macros of quantifiers such as $\forall x \exists y$ are considered [2].

The solution is to consider sequent calculi with concepts of proof which are globally but not locally sound, similar to [1]. This means that all derived statements are true but that not every sub-derivation is meaningful. We obtain for **LF** and **LH** globally, but possibly locally unsound calculi $\mathbf{LF}^{++}$ and $\mathbf{LH}^{++}$ by weakening the eigenvariable conditions and show soundness, completeness and cut-elimination for the novel calculus $\mathbf{LH}^{++}$ [3]. The main results are[4]:

Lemma 1. *An $\mathbf{LH}^{++}$-derivation φ with cuts can be immediately transformed into an $\mathbf{LF}^{++}$-derivation φ' with cuts.*

Lemma 2. *An $\mathbf{LF}^{++}$-derivation φ where the end-sequent contains only quantifiers in blocked distinct sequences $\exists f \exists g \forall x \forall y$ can be transformed into a cut-free $\mathbf{LF}^{++}$-derivation φ' where the quantifiers in the sequence $\exists f \exists g \forall x \forall y$ belonging to a block in the end-sequent are inferred immediately one after the other.*

Lemma 3. *A cut-free $\mathbf{LF}^{++}$-proof φ with blocked quantifier inferences $\exists f \exists g \forall x\, \forall y$ from atomic axioms and only such blocks of quantifiers in the end-sequent can be transformed into a cut-free $\mathbf{LH}^{++}$-proof φ' from atomic axioms.*

Theorem 1. $\mathbf{LH}^{++}$ *is sound, cut-free complete w.r.t. the intended semantics and admits an effective cut-elimination.*

It is obvious that the methodology developed in this work can be extended to arbitrary Henkin quantifiers, however not to arbitrary macros of quantifiers, where repeated alternations between strong and weak quantifiers are allowed.

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[3] The usual quantifier rules of **LK** ($\forall_l$,$\forall_r$ and $\exists_l$, $\exists_r$) can be obtained by partial dummy applications of Q_H.
[4] All proofs can be found in [3].

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