LINDSTRÖM THEOREM FOR PREDICATE INTUITIONISTIC LOGIC

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The talk will be devoted to the explanation of the main result of [4] (joint work with G. Badia and R. Zoghifard). The paper extends the main result of [1] to several variants of first-order intuitionistic logic. More precisely, we consider the following family of six logics which we call standard intuitionistic logics, $StIL$ for short:

- Intuitionistic first-order logic without equality;
- Intuitionistic first-order logic with extensional equality;
- Intuitionistic first-order logic with intensional equality;
- Intuitionistic logic of constant domains without equality;
- Intuitionistic logic of constant domains with extensional equality;
- Intuitionistic logic of constant domains with intensional equality.

We supply these logics with the semantics in style of 'modified Kripke semantics' of [2, Sect 5.3] and define, for every system in $StIL$, an appropriate intuitionistic variant of first-order bisimulation (initially introduced in [3] for the intuitionistic first-order logic without equality under the name of first-order asimulation).

We then define the notion of abstract intuitionistic logic $\mathcal{L}$ as a quadruple of the form $(Str_{\mathcal{L}}, L, \models_{\mathcal{L}}, \boxplus_{\mathcal{L}})$, where $Str_{\mathcal{L}}$ is a function returning, for every signature Θ, the class of $\mathcal{L}$-admissible pointed Θ-models $Str_{\mathcal{L}}(\Theta)$ and L is a function returning the set $L(\Theta)$ of Θ-sentences in $\mathcal{L}$; next, $\models_{\mathcal{L}}$ is a class-relation such that, if $\alpha \models_{\mathcal{L}} \beta$, then there exists a signature Θ such that $\alpha \in Str_{\mathcal{L}}(\Theta)$, and $\beta \in L(\Theta)$; informally this is to mean that β holds in α. The relation $\models_{\mathcal{L}}$ is only assumed to be defined (i.e. to either hold or fail) for the elements of the class $\bigcup\{(Str_{\mathcal{L}}(\Theta), L(\Theta)) \mid \Theta \text{ is a signature}\}$ and to be undefined otherwise.

The fourth element in our quadruple, $\boxplus_{\mathcal{L}}$, is then a function, returning, for every $(\mathcal{M}, w) \in Str_{\mathcal{L}}(\Theta)$, for every tuple $\bar{c}$ of pairwise distinct constants outside Θ, and for every tuple $\bar{a}$ of objects in $(\mathcal{M}, w)$ a non-empty set of admissible constant expansions of a certain family of submodels of $\mathcal{M}$ by this new tuple of constants in such a way that $\bar{a}$ ends up being exactly the tuple of their values at $(\mathcal{M}, w)$.

In order for such a quadruple $\mathcal{L}$ to count as an abstract intuitionistic logic, several groups of additional requirements need to be satisfied, and we will offer a detailed formulation and motivation of these requirements in the talk.

We further define that, given a pair of abstract intuitionistic logics $\mathcal{L}$ and $\mathcal{L}'$, we say that $\mathcal{L}'$ *expressively extends* $\mathcal{L}$ and write $\mathcal{L} \sqsubseteq \mathcal{L}'$ iff all of the following holds:

- $Str_{\mathcal{L}} = Str_{\mathcal{L}'}$;

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- If $(\mathcal{M}, w) \in Str_{\mathcal{L}}(\Theta)$, $\bar{c}$ is a tuple of pairwise distinct constants outside Θ, and $\bar{a}$ is an appropriate tuple of objects, then we have $(\mathcal{M}, w)\boxplus_{\mathcal{L}'}(\bar{c}_n/\bar{a}_n) \subseteq (\mathcal{M}, w)\boxplus_{\mathcal{L}}(\bar{c}_n/\bar{a}_n)$
- For every $\phi \in L(\Theta)$ there exists a $\psi \in L'(\Theta)$ such that for every $(\mathcal{M}, w) \in Str_{\mathcal{L}}(\Theta)$ we have:

$$\mathcal{M}, w \models_{\mathcal{L}} \phi \Leftrightarrow \mathcal{M}, w \models_{\mathcal{L}'} \psi.$$

If both $\mathcal{L} \sqsubseteq \mathcal{L}'$ and $\mathcal{L}' \sqsubseteq \mathcal{L}$ hold, then we say that the logics $\mathcal{L}$ and $\mathcal{L}'$ are *expressively equivalent* and write $\mathcal{L} \bowtie \mathcal{L}'$.

We show how the logics in $StIL$ can be represented as abstract intuitionistic logics in the form outlined above and consider abstract extensions of these logics. Some of these extensions enjoy useful model-theoretic properties among which the following three are of particular interest: (1) Compactness, (2) Tarski Union Property, and (3) preservation under $\mathcal{L}$-asimulations for some $\mathcal{L} \in StIL$.

Our main result is then that no standard intuitionistic logic has proper extensions that display the combination of all the three useful properties mentioned above. In other words, we establish the following:

Theorem 1. *Let $\mathcal{L}$ be an abstract intuitionistic logic and let $\mathcal{L}' \in StIL$. If $\mathcal{L}' \sqsubseteq \mathcal{L}$ and $\mathcal{L}$ is preserved under $\mathcal{L}'$-asimulations, compact, and has Tarski Union Property, then $\mathcal{L}' \bowtie \mathcal{L}$.*

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References

[1] G. Badia and G. Olkhovikov. A Lindström theorem for intuitionistic propositional logic. *Notre Dame Journal of Formal Logic*, 61 (1): 11–30 (2020).
[2] D. van Dalen. *Logic and Structure*. 4 ed. Springer, 2008.
[3] G. K. Olkhovikov. Model-theoretic characterization of intuitionistic predicate formulas. *Journal of Logic and Computation*, 24: 809–829 (2014).
[4] G. Olkhovikov, G. Badia and R. Zoghifard. A Lindström theorem for intuitionistic first-order logic. (submitted), preprint available at http://arxiv.org/abs/2103.17024.