The Distributive Full Lambek Calculus with Modal Operators: Discete duality and Kripke completeness

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In this talk, we study bounded distributive residuated lattices with modal operators $\Box$ and $\Diamond$ and their logics. We show that any canonical logic is Kripke complete via discrete duality and canonical extensions. We show that a given canonical modal extension of the distributive full Lambek calculus is the logic of its frames if its variety is closed under canonical extensions.

By the distributive full Lambek calculus with modal operators we mean the logic of the following kind:

Definition 1. A residual normal distributive modal logic is the set of sequents Λ that contains axioms (1)-(14) and closed under inference rules below:

$$p \Rightarrow p$$

- $\bot \Rightarrow p$
- $p \Rightarrow \top$
- $p_i \Rightarrow p_1 \vee p_2, i = 1, 2$
- $p_1 \wedge p_2 \Rightarrow p_i, i = 1, 2$
- $p \wedge (q \vee r) \Rightarrow (p \wedge q) \vee (p \wedge r)$
- $p \bullet (q \bullet r) \Leftrightarrow p \bullet (q \bullet r)$
- From $\varphi \Rightarrow \psi$ and $\psi \Rightarrow \theta$ infer $\varphi \Rightarrow \theta$
- From $\varphi \Rightarrow \psi$ and $\theta \Rightarrow \psi$ infer $\varphi \vee \theta \Rightarrow \psi$
- From $\varphi \bullet \theta \Rightarrow \psi$ infer $\theta \Rightarrow \varphi \setminus \psi$ and vice versa
- $p \bullet \mathbf{1} \Leftrightarrow \mathbf{1} \bullet p \Leftrightarrow p$
- $\Diamond(p \vee q) \Leftrightarrow \Diamond p \vee \Diamond q$
- $\Diamond\bot \Rightarrow \bot$
- $\Box p \wedge \Box q \Leftrightarrow \Box(p \wedge q)$
- $\top \Rightarrow \Box\top$
- $\Box p \bullet \Box q \Rightarrow \Box(p \bullet q)$
- From $\varphi(p) \Rightarrow \psi(p)$ infer $\varphi[p := \psi] \Rightarrow \psi[p := \gamma]$
- From $\varphi \Rightarrow \psi$ and $\varphi \Rightarrow \theta$ infer $\varphi \Rightarrow \psi \wedge \theta$
- From $\theta \bullet \varphi \Rightarrow \psi$ infer $\theta \Rightarrow \psi / \varphi$, and vice versa

In fact, a residual normal distributive modal logic extends normal distributive normal modal logic, the logic of bounded distributive lattices with modal operators introduced in [6]. To define relational semantics we introduce ternary Kripke frames with the additional binary modal relations. Such a ternary frame might be considered as a noncommutative generalisation of a modal relevant Kripke frame described, e.g., here [10]. As it is usual in the relational semantics of substructural logic, product and residuals have the ternary semantics as in, e.g., [1].

Definition 2. A Kripke frame is a tuple $\mathcal{F} = \langle W, R, R_\Box, R_\Diamond, \mathcal{O} \rangle$, where $R \subseteq W^3$, $R_\Box, R_\Diamond \subseteq W^2$, $\mathcal{O} \subseteq W$.

Note that R, $R_\Box$, and $R_\Diamond$ have certain conditions that we define in more detail during our talk. A Kripke model is a Kripke frame with an equipped valuation function that maps

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each propositional variable to $\leq$-upwardly closed subset of worlds. Let $\mathcal{F} = \langle W, R, R_\Box, R_\Diamond, \mathcal{O}\rangle$ be a Kripke frame, a Kripke model is a pair $\mathcal{M} = \langle \mathcal{F}, \vartheta\rangle$, where $\vartheta : \mathrm{PV} \to \mathrm{Up}(W, \leq)$. Here, $\mathrm{Up}(W, \leq)$ is the collection of all upwardly closed sets. Variables, $\wedge$, $\vee$, $\bot$, and $\top$ are understood standardly. The truth conditions for product, residuals, and $\mathbf{1}$ are understood with a ternary relation and the distinguished subset $\mathcal{O}$. $\Box$ and $\Diamond$ are understood as usual Kripkean necessity and possibility defined in terms of $R_\Box$ and $R_\Diamond$ relations.

The soundness theorem is formulated and proved in stadardly.

Theorem 1. *Let* $\mathbb{F}$ *be a class of Kripke frames, then* $\mathrm{Log}(\mathbb{F}) = \{\varphi \Rightarrow \psi \mid \mathbb{F} \models \varphi \Rightarrow \psi\}$ *is a residual distributive normal modal logic.*

Now we define algebraic semantics for such logics. The underlying algebraic structure for us is a residuated lattice [7]. A residuated lattice is called bounded distributive if its lattice reduct is bounded distributive. A residuated lattice morphism is a map $f : \mathcal{L}_1 \to \mathcal{L}_2$ that commutes with all operations. A residuated distributive modal algebra is a distributive bounded residuated lattice extended with normal modal operators $\Box$ and $\Diamond$ that distribute over finite infima and suprema correspondingly. One may also consider such algebras as full Lambek algebras [8] [9] lattice reducts of which are bounded distributive lattices. Note that we also require that $\Box$ is also "normal" with respect to a product. Such a "normality" corresponds to the promotion principle widely used in linear logic. This "normality" requirement is introduced as the additional inequation, more precisely:

Definition 3. A residuated distributive modal algebra (RDMA) is an algebra $\mathcal{M} = \langle \mathcal{R}, \Box, \Diamond\rangle$ with the following conditions for each $a, b \in \mathcal{R}$:

1. $\Box(a \wedge b) = \Box a \wedge \Box b$, $\Box\top = \top$
2. $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$, $\Diamond\bot = \bot$
3. $\Box a \cdot \Box b \leq \Box(a \cdot b)$

A RDMA-morphism is a residuated lattice morphism $f : \mathcal{M}_1 \to \mathcal{M}_2$ such that $f(\Box a) = \Box(f(a))$ and $f(\Diamond a) = \Diamond(f(a))$.

One may associate with an arbitrary residual normal modal logic its variety as follows:

Definition 4. Let $\mathcal{L}$ be a residual normal modal logic, then $\mathcal{V}_\mathcal{L}$ is a variety defined by the set unequations $\{\varphi \leq \psi \mid \mathcal{L} \vdash \varphi \Rightarrow \psi\}$

The usual Lindenbaum-Tarski construction provides us algebraic completeness for each residual distributive normal modal logic.

Theorem 2. *Let* $\mathcal{L}$ *be a residual normal modal logic, then there exists an RDMA* $\mathcal{M}_L$ *such that* $\mathcal{L} \vdash \varphi \Rightarrow \psi$ *iff* $\mathcal{R}_L \models \varphi \leq \psi$

Now we define completely distributive residuated perfect lattice as a distributive version of residuated perfect one defined in [2].

Definition 5. A distributive residuated lattice $\mathcal{L} = \langle L, \bigvee, \bigwedge, \cdot, \backslash, /, \varepsilon\rangle$ is called perfect distributive residuated lattice, if:

- $\mathcal{L}$ is a perfect distributive lattice

- $\cdot$, $\backslash$, and $/$ are binary operations on L such that $/$ and $\backslash$ right and left residuals of $\cdot$, repsectively. $\cdot$ is a complete operator on L, and $/ : L \times L^\delta \to L$, $\backslash : L^\delta \times L \to L$ are complete dual operators.

Here we formulate canonical extensions for bounded distributive lattices with a residuated family in the fashion of [3]. Note that one may provide canonical extensions for Heyting algebras similarly as it is described here [4]. See this paper to get acquainted with canonical extensions for bounded distributive lattices with operators closely [5].

Lemma 1. *Let $\mathcal{L} = \langle L, \cdot, \backslash, /, \varepsilon \rangle$ be a bounded distributive residuated lattice, then so $\mathcal{L}^\sigma = \langle L^\sigma, \cdot^\sigma, \backslash^\pi, /^\pi \rangle$ is. Moreover, $\mathcal{L}^\sigma$ is a perfect residuated distributive lattice.*

Definition 6. Let $\mathcal{L}$ be a perfect distributive residuated lattice and $\Box, \Diamond$ unary operators on $\mathcal{L}$, then $\mathcal{M} = \langle \mathcal{L}, \Box, \Diamond \rangle$ is called a perfect distributive residuated modal algebra, if

- $\Box \bigwedge A = \bigwedge \{\Box a \mid a \in A\}$
- $\Diamond \bigvee A = \bigvee \{\Diamond a \mid a \in A\}$
- $\Box a \cdot \Box b \leq \Box(a \cdot b)$

where $A \subseteq \mathcal{L}$

Given $\mathcal{M}$, $\mathcal{N}$ perfect distributive modal algebras, a map $\mathcal{M} \to \mathcal{N}$ is a homomorphism, if f is a complete lattice homomorphism that preserves product, residuals, modal operators, and the multiplicative identity.

Let us show that the variety of all RDMA is closed under canonical extensions.

Lemma 2. *Let $\mathcal{M}$ be a RDMA, then $\mathcal{M}^\sigma$ is a perfect DRMA, where $\mathcal{M}^\sigma$ is a canonical extension of the undelying bounded distributive residuated lattice with extended $\Box$ and $\Diamond$.*

Definition 7. A residual normal modal logic $\mathcal{L}$ is called canonical, if $\mathcal{V}_\mathcal{L}$ is closed under canonical extensions

Given a Kripke frame $\mathcal{F} = \langle W, R, R_\Box, R_\Diamond, \mathcal{O} \rangle$, we construct a complex algebra $\mathcal{F}^+$ as defining operations and constants $\mathrm{Up}(W, \leq)$. It is clear that $\bot = \emptyset$, $\top = W$, $\mathbb{1} = \mathcal{O}$, $A \wedge B = A \cap B$, $A \vee B = A \cup B$. Residuals, product, and modal operators are obtained via ternary and binary modal relations.

Let us define $\mathcal{M}_+$, the dual Kripke frame a perfect RDMA $\mathcal{M}$. Let us define the following relations on $\mathcal{J}^\infty(\mathcal{M})$: $aR_\Diamond b \Leftrightarrow b \leq \Diamond a$, $aR_\Box b \Leftrightarrow \Box\kappa(a) \leq \kappa(b)$, and $Rabc \Leftrightarrow a \cdot b \leq c$. The structure $\mathcal{M}_+ = \langle \mathcal{J}^\infty(\mathcal{M}), \leq, R, R_\Diamond, R_\Box, \mathcal{O} \rangle$ is the dual frame of a perfect RDMA $\mathcal{M}$, where $\mathcal{O} = \uparrow \{\varepsilon\}$.

Here, $\mathcal{J}^\infty(\mathcal{M})$ is the set of all completely join irreducible elements of a perfect RDMA $\mathcal{R}$ and κ is the isomorphism between the set of all completely join irreducible elements and the set of all completely meet irreducible elements defined as $a \mapsto \bigvee(- \uparrow a)$.

Lemma 3.

1. *A complex algebra of a Kripke frame $\mathcal{F}$ defined as $\mathcal{F}^+ = \langle \mathrm{Up}(W, \leq), \wedge, \vee, \bot, \top, \backslash, /, \cdot, \mathcal{O}, [R_\Box], \langle R_\Diamond \rangle, \rangle$ is a perfect DRMA.*
2. *Let $\mathcal{M}$ be a perfect DRMA, then $\mathcal{M}_+$ is a Kripke frame*

Theorem 3.

1. *Let* $\mathcal{F} = \langle W, R, R_\Box, R_\Diamond, \mathcal{O} \rangle$ *be a Kripke frame, then* $\mathcal{F} \cong (\mathcal{F}^+)_+$
2. *Let* $\mathcal{M} = \langle M, \bigvee, \bigwedge, \Box, \Diamond, \varepsilon \rangle$ *be a perfect DRMA, then* $\mathcal{R} \cong (\mathcal{M}_+)^+$
3. *Functors* $(.)_+ : \mathrm{pDRMA} \rightleftarrows \mathrm{KF} : (.)^+$ *establish a dual equivalence between the category of Kripke frames and the category of perfect DRMAs.*

The discrete duality established above together with canonical extensions of residuated distributive modal algebras provides the following consequence:

Theorem 4. *Let* $\mathcal{L}$ *be a canonical residual distributive modal logic, then* $\mathcal{L}$ *is Kripke complete.*

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