On relationship between complexity function and complexity of validity in propositional modal logic

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Abstract

We prove the existence, for any complexity class or degree of unsolvability C, of a linearly approximable extension of the unimodal propositional logic **K** whose variable-free fragment is C-hard. A similar result is proven for extensions of the unimodal propositional logic **KTB**.

1 Introduction

The study of computational properties of propositional modal, and related, logics has been historically concerned with estimating the size of smallest Kripke frames separating formulas from logics. The function f_L estimating, for a logic L, the size of smallest L-frames refuting L-inconsistent formulas is called (see e.g. [3, Chapter 18]) the *complexity function of L*. The interest in the complexity function of a logic L is largely due to it giving us an estimate of the running time of a decision algorithm for L-validity. In particular, provided the Kripke semantics for L is reasonably "natural," polynomiality of f_L implies that L can be shown to be polynomially equivalent to the classical propositional logic **Cl**—in the sense that the complexity of L-validity is the same as the complexity of **Cl**-validity modulo a polynomial—using a natural construction originally proposed by A. Kuznetsov [6] for the propositional intuitionistic logic **Int**, but subsequently adapted to propositional modal logics whose semantics can be described using classical propositional formulas [3, §18.1].[1]

The existence or otherwise of an inherent link between the nature of the complexity function of L and the complexity of L-validity is a natural question that, as far as we know, has not been explicitly considered in the literature. The existence, or at least plausibility, of such a link seems to be an underlying assumption in Kuznetsov's work [6]. The impression that such a link is plausible might well arise from similarities in the constructions used in establishing PSPACE-hardness of **Int** [15, 18] and the minimal normal modal logic **K** [7], on the one hand, and those used in proving the exponentiality of the complexity function of **Int** [19], [3, §18.2], [18] and **K** [1, §6.7], on the other.

A. Urquhart [16] has shown that, in propositional modal logic, enjoyment of the finite model property is compatible with undecidability. Building on Urquhart's work, E. Spaan [14,

[1]Perhaps the most widely known example of a "natural" propositional modal logic—i.e., one not purposefully constructed to exhibit a logic with a sought property—whose complexity function is polynomial but that is not polynomially equivalent to **Cl** unless NP = PSPACE is the linear-time temporal logic **LTL** [13]; the semantics of **LTL**, however, involves evaluating formulas with respect to paths rather than worlds, which precludes a straightforward application of Kuznetsov's construction.

Theorem 2.1.1] has shown that even the polynomial-size model property is compatible with undecidability; in fact, Spaan has shown that this is so even for single-variable fragments of extensions of **K**.

We further extend Spaan's results, in three respects: we show that similar connections hold, first, for arbitrary complexity classes or degrees of unsolvability, second, for even variable-free fragments, and third, for the logics higher up in the lattice of the normal modal logics (namely, extensions of **KTB**).

2 Preliminaries

We consider a propositional modal language containing a single unary modal operator $\Box$. We use the standard terminology and notation related to modal logic, as can be found in [3] and [1]. We use $\mathit{NExt}\,L$ to denote the lattice of normal extensions of the propositional modal logic L.

Recall that a propositional modal logic L is said to have the finite model property (fmp) if every L-consistent formula is satisfiable in a finite model based on an L-frame (equivalently, every formula not in L is refuted in a finite model based on an L-frame).

Given a logic L that has the fmp, the *complexity function of* L (see, e.g., [3, §18.1]) is defined by

$$f_L(n) \quad = \quad \max\big\{ \min\{|\mathfrak{F}| : \mathfrak{F} \models L,\, \mathfrak{F} \not\models \varphi\} : |\varphi| \leqslant n \text{ and } \varphi \notin L \big\},$$

where $|\mathfrak{F}|$ is the cardinality of a frame $\mathfrak{F}$ and $|\varphi|$ is the size of a formula φ. A logic L is *polynomially* (respectively, *linearly*) *approximable*, if there exists a positive constant c such that $f_L(n) \leqslant n^c$ (respectively, $f_L(n) \leqslant c \cdot n$), for sufficiently large n.

3 Main results

Given $n \geqslant 2$, let $\mathfrak{F}_n = \langle W_n, R_n\rangle$ be a Kripke frame where $W_n = \{w_0, \dots, w_n, w^*\}$ and $R_n = \{\langle w_k, w_{k+1}\rangle : 0 \leqslant k < n\} \cup \{\langle w_0, w^*\rangle\}$. Given $n \geqslant 1$, define $\alpha_n = \Diamond\Box\bot \wedge \Diamond^n\Box\bot$.

Lemma 3.1. *Let $m, k > 2$. Then, $\mathfrak{F}_m, x \models \alpha_k$ if, and only if, $k = m$ and $x = w_0$.*

Let $\mathbb{A} = \mathbb{N} \setminus \{0, 1\}$. Given a set $I \in 2^{\mathbb{A}}$, define $2 \cdot I = \{2n : n \in I\}$, $\mathfrak{C}_I = \{\mathfrak{F}_n : n \in \mathbb{A} \setminus 2 \cdot I\}$, and $L_I = L(\mathfrak{C}_I)$. Lemma 3.1 immediately gives us the following:

Lemma 3.2. *For every $n \in \mathbb{A}$,*

$$\neg\alpha_{2n} \in L_I \quad \Longleftrightarrow \quad \mathfrak{F}_{2n} \notin \mathfrak{C}_I \quad \Longleftrightarrow \quad n \in I.$$

Thus, L_I-validity is as hard as the decision problem for the set I. Therefore, given any complexity class or degree of unsolvability C, with an appropriate choice of I, we obtain a C-hard normal modal logic L_I—in fact, since formulas α_n contain no propositional variables, even the variable-free fragment of L_I is C-hard.[2]

Lemma 3.3. *For every $I \subseteq \mathbb{A}$, the logic L_I is linearly approximable.*

Lemmas 3.3 and 3.2 give us the following:

Theorem 3.4. *Let C be a complexity class or a degree of unsolvability. Then, there exists a linearly approximable logic $L \in \mathit{NExt}\,\mathbf{K}$ whose constant fragment is C-hard.*

[2]For many "natural" modal logics, the complexity of their variable-free fragments coincides with the complexity of the full logic [2, 9].

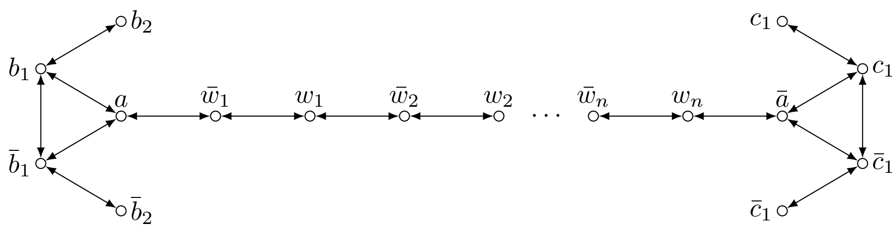

Figure 1: Frame $\mathfrak{F}_n^{rs}$

Similar examples can be constructed higher up in the lattice of the normal propositional modal logics. We illustrate this claim with extensions of the logic of reflexive and symmetric frames **KTB**.

For every $n \geqslant 1$, let $\mathfrak{F}_n^{rs} = \langle W_n, R_n \rangle$ be a Kripke frame (see Figure 1) where

$$W_n \quad = \quad \{w_1, \dots, w_n\} \cup \{\overline{w}_1, \dots, \overline{w}_n\} \cup \{a, \overline{a}, b_1, \overline{b}_1, b_2, \overline{b}_2, c_1, \overline{c}_1, c_2, \overline{c}_2\}$$

and R_n is the reflexive and symmetric closure of the relation

$$\begin{array}{l}\{\langle \overline{w}_k, w_k \rangle : 1 \leqslant k \leqslant n\} \ \cup \{\langle a, \overline{w}_1, \langle \overline{a}, w_n \rangle\} \cup \\ \{\langle a, b_1 \rangle, \langle a, \overline{b}_1 \rangle, \langle b_1, b_2 \rangle, \langle \overline{b}_1, \overline{b}_2 \rangle, \langle b_1, \overline{b}_1 \rangle\} \ \cup \{\langle \overline{a}, c_1 \rangle, \langle \overline{a}, \overline{c}_1 \rangle, \langle c_1, c_2 \rangle, \langle \overline{c}_1, \overline{c}_2 \rangle, \langle c_1, \overline{c}_1 \rangle\}.\end{array}$$

Recursively define the sequence of formulas

$$\begin{array}{lcl}\zeta_0 & = & \neg p \wedge \Diamond^{=2}\Box p \wedge \Diamond^{=2}\Box \neg p; \\ \zeta_{k+1} & = & \neg p \wedge \Diamond(p \wedge \Diamond \zeta_k),\end{array}$$

and define, for every $n \geqslant 1$ (letting $\Diamond^{=2}\varphi = \Diamond\Diamond\varphi \wedge \neg\Diamond\varphi$),

$$\gamma_n \quad = \quad p \wedge \Diamond^{=2}\Box p \wedge \Diamond^{=2}\Box\neg p \wedge \Diamond\zeta_n \wedge \bigwedge_{k=0}^{n-1} \neg\Diamond\zeta_k.$$

Lemma 3.5. *Let $m, k > 2$ and let x be a world in $\mathfrak{F}_m^{rs}$. Then, γ_k is satisfiable at x if, and only if, $k = m$ and $x \in \{a, \overline{a}\}$.*

Let $\mathfrak{C}_I^{rs} = \{\mathfrak{F}_n^{rs} : n \in \mathbb{A} \setminus 2 \cdot I\}$ and $L_I^{rs} = L(\mathfrak{C}_I^{rs})$. Lemma 3.5 immediately gives us the following:

Lemma 3.6. *For every $n \in \mathbb{A}$,*

$$\neg\gamma_{2n} \in L_I^{rs} \quad \Longleftrightarrow \quad \mathfrak{F}_{2n}^{rs} \notin \mathfrak{C}_I^{rs} \quad \Longleftrightarrow \quad n \in I.$$

Therefore, L_I^{rs}-validity is as hard as the decision problem for the set I. Consequently, given any complexity class or degree of unsolvability C, with an appropriate choice of I, we obtain a C-hard normal extension L_I^{rs} of **KTB**—in fact, since formulas γ_n contain only one propositional variable, even the single-variable fragment of L_I^{rs} is C-hard.[3]

Lemma 3.7. *For every $I \subseteq \mathbb{N}^+$, the logic L_I^{rs} is linearly approximable.*

Lemmas 3.7 and 3.6 give us the following:

Theorem 3.8. *Let C be a complexity class or a degree of unsolvability. Then, there exists a linearly approximable logic $L \in NExt$* **KTB** *whose single-variable fragment is C-hard.*

[3]For many "natural" modal logics, the complexity of their single-variable fragments coincides with the complexity of the full logic [5, 4, 2, 17, 8, 10, 11, 12].

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