Boxing modal logics

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We consider normal 1-modal logics, propositional and predicate. For the basic definitions cf. [1], [2].

1 Propositional logics

For a set of modal formulas Γ, put

$$\Box\Gamma := \{\Box A \mid A \in \Gamma\}.$$

For a modal propositional logic L put

$$\Box{\cdot}L := \mathbf{K} + \Box L.$$

Lemma 1.1. $\Box{\cdot}(\mathbf{K} + \Gamma) = \mathbf{K} + \Box\Gamma$.

It turns out that $\Box{\cdot}L$ inherits many properties of L.

Theorem 1.2.

- *If L is Kripke complete, then $\Box{\cdot}L$ is Kripke complete.*
- *If L is strongly Kripke complete, then $\Box{\cdot}L$ is strongly Kripke complete.*
- *If L is canonical, then $\Box{\cdot}L$ is canonical.*
- *If L has the FMP, then $\Box{\cdot}L$ has the FMP.*
- *If L is locally tabular, then $\Box{\cdot}L$ is locally tabular.*
- *If L is has a finite modal depth, then $\Box{\cdot}L$ has a finite modal depth:*

$$md(\Box{\cdot}L) \leq md(L) + 1.$$

Hence, in particular, we obtain many new examples of locally tabular logics.

Corollary 1.3. *The logics $\mathbf{K} + \Box^n(p \to \Box p)$ (and all their extensions) are locally tabular.*

Another consequence is the FMP for some logics of trees. Recall that a *tree* (irreflexive and intransitive) is a rooted frame, in which every point (but the root) has a unique predecessor. A *reflexive tree* is a reflexive closure of a tree.

Theorem 1.4. *The logic of every serial tree has the FMP.*

Theorem 1.5. *The logic of every reflexive tree has the FMP.*

Theorem 1.6. *The logic of every tree validating*

$$\Diamond\top \to \Diamond^2\top \wedge \Diamond\Box\bot$$

has the FMP.

2 Predicate logics

Recall that $\mathbf{Q\Lambda}$ is the minimal predicate extension of a propositional logic $\mathbf{\Lambda}$; $\mathbf{T} = \mathbf{K} + \Box p \to p$.

For a predicate modal logic L we also define boxing:

$$\Box\cdot L := \mathbf{QK} + \Box L.$$

For modal predicate logics a direct analogue of Lemma 1.1 does not hold. It is replaced by the following

Lemma 2.1. $\Box\cdot(\mathbf{QT} + \Gamma) = \mathbf{QT} + \Box\Gamma + \Box\forall ref$, *where*

$$\Box\forall ref := \Box\forall x(\Box P(x) \to P(x)).$$

Axiomatization of boxing in other cases remains an open problem.

Definition 2.2. *A* predicate modal theory Γ *is a set of closed predicate modal formulas with constants.*

A predicate modal theory Γ *is* satisfiable *in a predicate Kripke frame* $\mathbf{F}$ *if there exists a Kripke model* M *over* $\mathbf{F}$, *a world* w *in* M *and a map* δ *from constants of* Γ *to the domain of* w *such that* $M, w \vDash \delta \cdot \Gamma$.

A predicate modal logic L *is* strongly Kripke complete *if every* L*-consistent countable theory* Γ *is satisfiable in a Kripke frame validating* L.

Theorem 2.3. *Let* $\mathbf{\Lambda}$ *be a modal propositional logic containing* $\mathbf{T}$. *If* $\mathbf{Q\Lambda}$ *is strongly Kripke complete, then* $\Box\cdot\mathbf{Q\Lambda}$ *is strongly Kripke complete.*

There are several well-known examples of logics $\mathbf{\Lambda}$ above $\mathbf{T}$, for which $\mathbf{Q\Lambda}$ is strongly Kripke complete: **T**, **S4**, **S5**, **S4.2**, **S4.3**, **Triv**. So in these cases boxing preserves strong Kripke completeness.

The definition of strong completeness can be extended to Kripke sheaf semantics. Then we can prove a better result:

Theorem 2.4. *If a predicate modal logic* L *is strongly Kripke sheaf complete, then* $\Box\cdot L$ *is strongly Kripke sheaf complete.*

On the other hand, quite often logics of the form $\mathbf{QK} + \Box\Gamma$ are Kripke (and Kripke sheaf) incomplete. In particular, we have

Theorem 2.5. *If* $\mathbf{\Lambda}$ *is any consistent modal propositional logic containing* $\mathbf{T}$, *then* $\mathbf{Q}(\Box\cdot\mathbf{\Lambda})$ *is Kripke incomplete, and* $\Box\cdot(\mathbf{Q\Lambda}) = \mathbf{Q}(\Box\cdot\mathbf{\Lambda}) + \Box\forall ref$ *is its Kripke completion.*

References

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