On a Possibility of Finite Characterizations for Kripke Complete Non-Recursively Axiomatizable Superintuitionistic Predicate Logics

Dmitrij Skvortsov

It is well known that there exist many natural classes $\mathcal{F}$ of predicate Kripke frames for which the corresponding superintuitionistic predicate logics $\mathbf{L}[\mathcal{F}]$ are non-recursively axiomatizable (see e.g. [1, 2, 3]). Here we consider a possibility of finite semantical characterizations for some logics of this kind.

Denote the superintuitionistic predicate logic of a (predicate) Kripke frame F (i.e., the set of formulas valid in F) by $\mathbf{L}F$. Recall that a predicate logic $\mathbf{L}$ is called *Kripke complete* if $\mathbf{L} = \mathbf{L}[\mathcal{F}]$ for some class $\mathcal{F}$ of Kripke frames, where $\mathbf{L}[\mathcal{F}] = \bigcap(\mathbf{L}F : F \in \mathcal{F})$. The *Kripke completion* of a logic $\mathbf{L}$ is the smallest (w.r.t. the inclusion) Kripke complete extension of $\mathbf{L}$.

Let $\mathbf{L}$ and $\mathbf{L}_0$ be two predicate logics, $\mathbf{L}$ being Kripke complete. We say that $\mathbf{L}_0$ *semantically generates* $\mathbf{L}$ if $\mathbf{L}$ is the Kripke completion of $\mathbf{L}_0$.

A logic $\mathbf{L} = \mathbf{L}[\mathcal{F}]$ of a class $\mathcal{F}$ of Kripke frames is called *finitely* (or *recursively*) *semantically generated* if there exists a finitely (or, resp., recursively) axiomatizable logic $\mathbf{L}_0$ semantically generating $\mathbf{L}$. In this case we also can say that the class $\mathcal{F}$ itself is finitely (resp., recursively) semantically generated. This notion can be regarded as a semantical analogue to finite / recursive axiomatizability; namely, semantical comletion (i.e., 'semantical closure') is used instead of deductive closure (i.e., deductive consequence). Clearly, this notion can be interesting only for natural classes of Kripke frames.

Obviously, every finitely / recursively axiomatizable Kripke complete logic is trivially finitely / recursively semantically generated: just take $\mathbf{L}_0 = \mathbf{L}$. Here we show that natural Kripke complete non-recursively axiomatizable logics can be still finitely (hence, recursively) semantically generated.

Proposition. *Let* $\mathbf{L} = \mathbf{L}[\mathcal{F}]$ *be the logic of a class of Kripke frames* $\mathcal{F}$*, let* $\mathbf{L}_0$ *be a predicate logic, and let* $\mathcal{F}_{\mathbf{L}_0}$ *be the class of Kripke frames validating* $\mathbf{L}_0$*. Then:* $\mathbf{L}_0$ *semantically generates* $\mathbf{L}$ *iff the following two conditions hold:*

(*i*) $\mathcal{F} \subseteq \mathcal{F}_{\mathbf{L}_0}$ *or, equivalently,* $\mathbf{L}_0 \subseteq \mathbf{L}(= \mathbf{L}[\mathcal{F}])$*;*

(*ii*) $\mathbf{L} \subseteq \mathbf{L}F$ *for every frame* F *from* $\mathcal{F}_{\mathbf{L}_0} \backslash \mathcal{F}$*.*

Proof. Let $\mathbf{L}^* = \mathbf{L}[\mathcal{F}_{\mathbf{L}_0}]$ be the Kripke completion of $\mathbf{L}_0$; hence

$\mathbf{L}_0$ semantically generates $\mathbf{L}$ iff (0) $\mathbf{L}^* = \mathbf{L}$.

The condition (*ii*) means that $\mathbf{L} \subseteq \mathbf{L}[\mathcal{F}_{\mathbf{L}_0} \backslash \mathcal{F}]$.

Therefore, the condition (0) obviously implies (*ii*), as well as (*i*).

On the other hand, (*i*) implies that $\mathbf{L}^* = \mathbf{L} \cap \mathbf{L}[\mathcal{F}_{\mathbf{L}_0} \backslash \mathcal{F}]$, so (*ii*) gives (0). $\square$

Let $\mathbf{Fin}^c$ be the class of frames with constant domains over finite posets; let $\mathbf{P}^c_\infty$ be the class of frames of finite height with constant domains; finally, let $\mathbf{WF}^c_d$ be the class of frames with constant domains over dually well-founded (i.e., Nötherian) posets.

Theorem. *(The logics of) the classes of frames* $\mathbf{Fin}^c$, $\mathbf{P}^c_\infty$, $\mathbf{WF}^c_d$ *are finitely semantically generated.*

Cf. [4], Proposition 1 and footnote 9 from Sect. 3.2, and Lemma 3 from Sect. 4.3. Note that here the condition (*ii*) from our Proposition follows from the subsequent claim (cf. [4], Corollary 2 from Sect. 3.1):

Claim. $\mathbf{L}[\mathbf{Fin}^c] \subseteq \mathbf{L}F$ *for every frame F with a finite constant domain.*

We do not know if the logics of the corresponding classes of frames with expanding domains are finitely (or at least recursively) semantically generated.

References

[1] Skvortsov D. On axiomatizability of some intermediate predicate logics (summary). Reports on Math. Logic 22 (1988), pp. 115–116.

[2] Skvortsov D. The predicate logic of finite Kripke frames is not recursively axiomatizable. Journal of Symbolic Logic 70 (2005), pp. 451–459.

[3] Skvortsov D. On non-axiomatizability of superintuitionistic predicate logics of some classes of well-founded and dually well-founded Kripke frames. Journ. of Logic and Computation 16, No.5 [Special Issue: Computer Science Applications of Modal Logic (ed.V.B.Shehtman)] (2006), pp.685-695.

[4] Skvortsov D. A remark on the superintuitionistic predicate logic of Kripke frames of finite height with constant domains: A simpler Kripke complete logic that is not strongly complete. Advances in Modal Logic, Vol.12, College Publications, 2018, pp.577-590 [Intern. Conf. AiML 2018, 27 - 31 Aug. 2018, Bern, Switzerland].