Exactly True and Non-falsity versions of Deutsch's logic

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Abstract

In this report, we study logical systems which represent entailment relations of two kinds. We extend the approach of finding 'exactly true' and 'non-falsity' versions of four-valued logics that emerged in series of recent works on **FDE** to the case of infectious ones, namely to the case of Deutsch's relevant logic introduced in [8, 9].

A lot of interest was paid to so-called infectious logics in recent years. Besides their philosophical significance (see [19, 13]), a number of important results connected with applications of infectious logics in the context of the logical programming and proof theory were also obtained [5, 7, 6, 11, 18]. One interesting four-valued logic can be distinguished among this class of theories, namely Deutsch's $\mathbf{S_{fde}}$ [8, 9]. It can be seen as a rival of well-known Dunn-Belnap's four-valued logic **FDE** [10, 3, 4]. The difference lies in the interpretation of the truth value gaps, as is seen from the matrix below.

We fix a standard propositional language $\mathscr{L}$ with an alphabet $\langle \mathcal{P}, \sim, \wedge, \vee, (,) \rangle$, where $\mathcal{P} = \{p, q, r, s, p_1, \ldots\}$ is a set of propositional variables. The set $\mathscr{F}$ of all $\mathscr{L}$-formulas is defined in a standard inductive way. The set $\mathscr{V}_4 = \{\mathtt{T}, \mathtt{B}, \mathtt{N}, \mathtt{F}\}$ contains truth-values which are interpreted as follows: 'true', 'both' (i. e. both true and false), 'none' (i. e. neither true nor false), and 'false', respectively. A valuation is understood as a mapping from $\mathcal{P}$ to $\mathscr{V}_4$. It is extended on the set $\mathscr{F}$ according to the logical matrices which are presented below.

$\mathbf{S_{fde}}$ has the matrix $\langle \mathscr{V}_4, \sim, \wedge, \vee, \{\mathtt{T}, \mathtt{B}\} \rangle$, where:

φ	$\sim$
T	F
B	B
N	N
F	T

$\wedge$	T	B	N	F
T	T	B	N	F
B	B	B	N	F
N	N	N	N	N
F	F	F	N	F

$\vee$	T	B	N	F
T	T	T	N	T
B	T	B	N	B
N	N	N	N	N
F	T	B	N	F

The entailment relation is defined as preserving designated values.

Definition 1. For each $\Gamma \cup \Delta \subseteq \mathscr{F}$, it holds that:

- $\Gamma \models_{\mathbf{S_{fde}}} \Delta$ iff for each valuation v, $v(\gamma) \in \{\mathtt{T}, \mathtt{B}\}$ (for each $\gamma \in \Gamma$) implies $v(\delta) \in \{\mathtt{T}, \mathtt{B}\}$ (for some $\delta \in \Delta$);

In this work we introduce two new logics which differ from $\mathbf{S_{fde}}$ by the definition of the entailment relation. In a manner similar to what has been done by Kapsner[1] and Rivieccio in [14] and Shramko, Zaitsev and Belikov in [16, 17] regarding **FDE**, we consider the corresponding counterparts of $\mathbf{S_{fde}}$. The first one is $\mathbf{S_{etl}}$, the 'exactly true' version of $\mathbf{S_{fde}}$. It differs from $\mathbf{S_{fde}}$ by the set of designated values: it has $\{\mathtt{T}\}$ instead of $\{\mathtt{T}, \mathtt{B}\}$. The second one is $\mathbf{S_{nfl}}$, the 'non-falsity' version of $\mathbf{S_{fde}}$. It has the following set of designated values: $\{\mathtt{T}, \mathtt{B}, \mathtt{N}\}$.

Definition 2. For each $\Gamma \cup \Delta \subseteq \mathscr{F}$, it holds that:

[1]Pietz – before name changing.

- $\Gamma \models_{\mathbf{S_{etl}}} \Delta$ iff for each valuation v, $v(\gamma) = \mathtt{T}$ (for each $\gamma \in \Gamma$) implies $v(\delta) = \mathtt{T}$ (for some $\delta \in \Delta$);
- $\Gamma \models_{\mathbf{S_{nfl}}} \Delta$ iff for each valuation v, $v(\gamma) \in \{\mathtt{T}, \mathtt{B}, \mathtt{N}\}$ (for each $\gamma \in \Gamma$) implies $v(\delta) \in \{\mathtt{T}, \mathtt{B}, \mathtt{N}\}$ (for some $\delta \in \Delta$).

We provide a characterization of $\mathbf{S_{etl}}$ and $\mathbf{S_{nfl}}$ entailment relations with respect to the ones of $\mathbf{K_3}$ [12] and **LP** [15], respectively.

Theorem 1. *Let* $\Gamma \cup \Delta \subseteq \mathscr{F}$.
$\Gamma \models_{\mathbf{S_{etl}}} \Delta$ *iff* $\Gamma \models_{\mathbf{K_3}} \Delta'$ *for some* $\Delta' \subseteq \Delta$ *such that* $var(\Delta') \subseteq var(\Gamma)$.

Theorem 2. *Let* $\Gamma \cup \Delta \subseteq \mathscr{F}$.
$\Gamma \models_{\mathbf{S_{nfl}}} \Delta$ *iff* $\Gamma' \models_{\mathbf{LP}} \Delta$ *for some* $\Gamma' \subseteq \Gamma$ *such that* $var(\Gamma') \subseteq var(\Delta)$.

As to the main result, we introduce sound and complete Gentzen-style calculi (enjoying cut-elimination) for $\mathbf{S_{etl}}$ and $\mathbf{S_{nfl}}$. Consider the following set of axioms and sequent rules:

- Axioms:

  (Ax) $\varphi \Rightarrow \varphi$ (ECQ) $\varphi, {\sim}\varphi \Rightarrow$ (EM) $\Rightarrow \varphi, {\sim}\varphi$

- Structural rules:

  $$(\mathrm{W}{\Rightarrow})\ \frac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta} \qquad ({\Rightarrow}\mathrm{W})\ \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi} \qquad (\mathrm{Cut})\ \frac{\Gamma \Rightarrow \Delta, \varphi \quad \varphi, \Theta \Rightarrow \Pi}{\Gamma, \Delta \Rightarrow \Theta, \Pi}$$

- Logical rules:

$$(\wedge \Rightarrow)\ \frac{\varphi, \psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \wedge)\ \frac{\Gamma \Rightarrow \Delta, \varphi \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi}$$

$$(\vee \Rightarrow)\ \frac{\varphi, \Gamma \Rightarrow \Delta \quad \psi, \Gamma \Rightarrow \Delta}{\varphi \vee \psi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \vee)\ \frac{\Gamma \Rightarrow \Delta, \varphi, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}$$

$$({\sim}{\sim} \Rightarrow)\ \frac{\varphi, \Gamma \Rightarrow \Delta}{{\sim}{\sim}\varphi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow {\sim}{\sim})\ \frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, {\sim}{\sim}\varphi}$$

$$({\sim}\wedge \Rightarrow)\ \frac{{\sim}\varphi, {\sim}\psi, \Gamma \Rightarrow \Delta}{{\sim}(\varphi \wedge \psi), \Gamma \Rightarrow \Delta} \qquad (\Rightarrow {\sim}\wedge)\ \frac{\Gamma \Rightarrow \Delta, {\sim}\varphi \qquad \Gamma \Rightarrow \Delta, {\sim}\psi}{\Gamma \Rightarrow \Delta, {\sim}(\varphi \wedge \psi)}$$

$$({\sim}\vee \Rightarrow)\ \frac{{\sim}\varphi, \Gamma \Rightarrow \Delta \qquad {\sim}\psi, \Gamma \Rightarrow \Delta}{{\sim}(\varphi \vee \psi), \Gamma \Rightarrow \Delta} \qquad (\Rightarrow {\sim}\vee)\ \frac{\Gamma \Rightarrow \Delta, {\sim}\varphi, {\sim}\psi}{\Gamma \Rightarrow \Delta, {\sim}(\varphi \vee \psi)}$$

- The restricted versions of the logical rules:

$$(\wedge^{H} \Rightarrow)\ \frac{\varphi, \psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta} \quad \text{provided that } var(\{\varphi, \psi\}) \subseteq var(\Delta)$$

$$(\Rightarrow \vee^{B})\ \frac{\Gamma \Rightarrow \Delta, \varphi, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi} \quad \text{provided that } var(\{\varphi, \psi\}) \subseteq var(\Gamma)$$

Let us make some remarks regarding the rules and sequent calculi already mentioned in the literature. Let us write $\mathfrak{S}_{\mathbf{L}}$ for the sequent calculus for the logic **L**.

1. The axiom (Ax), all the structural rules, and the logical rules $(\wedge\Rightarrow)$, $(\Rightarrow\wedge)$, $(\vee\Rightarrow)$, $(\Rightarrow\vee)$, $({\sim}{\sim}\Rightarrow)$, $(\Rightarrow{\sim}{\sim})$, $({\sim}\wedge\Rightarrow)$, $(\Rightarrow{\sim}\wedge)$, $({\sim}\vee\Rightarrow)$, $(\Rightarrow{\sim}\vee)$ form the sequent calculus for **FDE** [1, 2].
2. The extension of $\mathfrak{S}_{\mathbf{FDE}}$ by the axiom (ECQ) is the sequent calculus for $\mathbf{K_3}$ [1].
3. The extension of $\mathfrak{S}_{\mathbf{FDE}}$ by the axiom (EM) is the sequent calculus for **LP** [1].

Let us extend this list by the new results.

4. The axioms (Ax) and (ECQ) as well as all the structural rules and the logical rules $({\sim}{\sim}\Rightarrow)$, $(\Rightarrow{\sim}{\sim})$, $(\wedge\Rightarrow)$, $(\Rightarrow\wedge)$, $(\vee\Rightarrow)$, $(\Rightarrow\vee^B)$, $({\sim}\wedge\Rightarrow)$, $(\Rightarrow{\sim}\wedge)$, $({\sim}\vee\Rightarrow)$, $(\Rightarrow{\sim}\vee)$ form the sequent calculus for $\mathbf{S_{etl}}$.
5. The axioms (Ax) and (EM) as well as all the structural rules and the logical rules $({\sim}{\sim}\Rightarrow)$, $(\Rightarrow{\sim}{\sim})$, $(\wedge^H\Rightarrow)$, $(\Rightarrow\wedge)$, $(\vee\Rightarrow)$, $(\Rightarrow\vee)$, $({\sim}\wedge\Rightarrow)$, $(\Rightarrow{\sim}\wedge)$, $({\sim}\vee\Rightarrow)$, $(\Rightarrow{\sim}\vee)$ form the sequent calculus for $\mathbf{S_{nfl}}$.

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