A joint logic of problems and propositions

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Abstract

We consider the joint logic of problems and propositions suggested by S. A. Melikhov. We prove that its propositional part is complete with respect to models constructed by S. Artemov and T. Protopopesku for intuitionistic epistemic logic. We also show that this logic conservatively extends the intuitionistic epistemic logic IEL^+.

In a commentary to his collected works [2], Kolmogorov remarked that his paper [3] «was written in hope that with time, the logic of solution of problems [i.e., intuitionistic logic] will become a permanent part of a [standard] course of logic. A unified logical apparatus was intended to be created, which would deal with objects of two types — propositions and problems.» Melikhov [5] construct a formal system QHC (a joint logic of problems and propositions) which contain two types of variables: problem and proposition. Formulas of QHC are built from variables by using standart classical and intuitionistic connectives $\vee, \wedge, \neg, \rightarrow$, modalities ! and ? and quantifiers $\forall, \exists$.

All propopositions (all problems) satisfy all inference rules and axioms schemes of classical (intuitionistic) predicate logic. Formulas of this two types are interconnected of two modalities. The modality ! inputs a proposition p and outputs a problem $!p$ «Find a proof of p». The modality ? inputs a problem α and outputs a proposition $?\alpha$ «There exists a solution of α». There are following axioms schemes and inference rules for modalities:

$$1)\ !(p \rightarrow q) \rightarrow (!p \rightarrow !q); \quad 2)\ ?(\alpha \rightarrow \beta) \rightarrow (?\alpha \rightarrow ?\beta);$$

$$3)\ \frac{p}{!p}; \quad 4)\ \frac{\alpha}{?\alpha};$$

$$5)\ ?!p \rightarrow p; \quad 6)\ \alpha \rightarrow !?\alpha; \quad 7)\ \neg !0.$$

Melikhov examined several types of models for logic QHC [6], but completeness theorem failed even for propositional part HC of this logic. Author [1] considered algebraical models and Kripke-type semantic for logic HC. Compleness theorem and finite model property are proved for this types of models.

Artemov and Protopopesku [4] considered audit set models for intuitionistic epistemic logic IEL^+.

Definition 1. *Audit set scale is a triplet $(W, \preccurlyeq, Aud)$, where $(W, \preccurlyeq)$ is a standard intuitionistic scale (W is a set, $\preccurlyeq$ is a partial order), $Aud \subseteq W$ is a subset of audit states such that*

$$\forall a \in W\ \exists b \in W (a \preccurlyeq b \wedge b \in Aud).$$

Let us define the evaluation $\models$ for intuitionistic formulas by standard way, for classical formulas by naturally way only in audit states, and for modalities this way:

$$a \models ?\alpha \Leftrightarrow a \models \alpha \text{ (for } a \in \mathrm{Aud})$$
$$a \models !p \Leftrightarrow \forall b \in \mathrm{Aud}(a \preccurlyeq b \Rightarrow b \models p) \text{ (for } a \in W),$$

We obtain a audit set model of logic HC.

Theorem 1. *Logic HC is complete with respect to audit set models. Moreover, the finite model property holds.*

Corollary 1. *Logic IEL^+ is complete with respect to finite audit set models.*

Corollary 2. *Logics HC and IEL^+ are decidable.*

Logic QHC is conservative extension of classical logic, intuitionistic logic and modal logic S4. This fact was proved by Melikhov. There was an open questions whether QHC is conservative extension of modal intuitionistic logic QH4 (let us denote $\nabla = !?$) which propositional part coincide which logic IEL^+. It is possible to extend audit set model of logic HC to models of logic QHC by attaching a set of «available objects» to each element of W. Author proved following theorem by using this models.

Theorem 2. *Logic QHC is conservative extension of logic QH4.*

References

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