Definability of graph properties in modal languages

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Graphs provide a descriptively effective theoretical framework for a lot of branches of mathematics and computer science. Despite the high level of generality from the descriptive point of view, graph theory lacks common methods that could be used to check arbitrary graph properties in a similar way. Usually each graph problem has to be solved and each graph property has to be tested with a specific method that usually does not generalize to other different problems or properties.

In the last few decades, logical languages, and particularly *modal languages*, have attracted the attention as the means for generalized solution of graph problems. Modal logics are preferred due to Kripke semantics that allow to evaluate modal formulas in structures that are essentially directed graphs. Graph properties can be defined in modal languages through the concept of *validity in a frame*. Given a graph G and a formula ϕ, we use the notation $G \models \phi$ for "ϕ is valid in G", considering a directed graph as a Kripke frame with a single relation defined by the edge set of G. The problem of determining if $G \models \phi$ for a given modal formula ϕ and a graph G, known as the *frame-checking problem*, is decidable. Its computational complexity is estimated in [3]: it is shown that the frame-checking problem is $PTIME$ (linear) in the length of formula and $EXPTIME$ in the size of frame.

There are several approaches to define a graph property with logical formulas.

1. A property is called finitely definable if there exists a single formula ϕ such that a graph G has the desired property if and only if $G \models \phi$.

2. A property is called definable if there exists a (possibly infinite) set of formulas Φ such that a graph G has the desired property if and only if $\forall \phi \in \Phi \ (G \models \phi)$.

3. A property is called co-definable if there exists a (possibly infinite) set of formulas Φ such that a graph G has the desired property if and only if $\exists \phi \in \Phi \ (G \models \phi)$.

In particular, a graph property is co-definable if there exists a countable set of formulas $\{\phi_n\}_{n \in \mathbb{N}}$ such that for each $n \in \mathbb{N}$ any graph G_n of cardinality n has the property is and only if $G_n \models \phi_n$. This important special case is a common means of defining complex graph properties such as being Hamiltonian.

A study of graph properties from the point of view of modal definability is presented in [1], [2], and [3]. In [1], where the problem of modal definability for graph properties was first stated, the author proves that the basic modal language fails to express several important properties, such as being connected, planar or Eulerian. Extensions of the modal language, including hybrid logics, are discussed in [3], and examples of formulas defining graph properties are given. These results are generalized in [2]: it is shown that every NP graph property is co-definable (with a countable set of formulas dependent on graph size) in the language $\mathcal{FHL}$ (Full Hybrid Logics).

The latter result shows the effectiveness of the chosen method but still can be improved and extended. First, the considered language is complex and therefore computationally non-optimal compared to the known algorithms for several graph properties. Second, it does not cover the important question of finite definability for graph properties.

In this talk we will discuss different approaches to defining graph properties, such as acyclicity, connectivity, planarity, etc., with the language of modal logics and its extensions. We consider the basic modal language $\mathcal{ML}$ with propositional variables, Boolean connectives and the modal operator $\Diamond$:

$$\mathcal{ML} ::= \top \mid p \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \Diamond\phi.$$

Then we define more expressive languages by enriching it with additional modal operators and symbols, including:

1. the inverse modality $\Diamond^{-1}$;
2. the transitive modality $\Diamond^{+}$;
3. the inverse transitive modality $\Diamond^{-}$;
4. the global modality E [4];
5. the nominals i and the satisfaction operators $@_i$ [4];
6. the state variables x and the bind operators $\downarrow x.\cdot$ [4].

We combine these extensions to define the extended modal languages as follows:

$$\begin{aligned}
\mathcal{ML}^{\pm} &::= \top \mid p \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \Diamond\phi \mid \Diamond^{+}\phi \mid \Diamond^{-1}\phi \mid \Diamond^{-}\phi;\\
\mathcal{PH} &::= \top \mid i \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \Diamond\phi;\\
\mathcal{PH}(@) &::= \top \mid i \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \Diamond\phi \mid @_i\phi;\\
\mathcal{PH}(\mathsf{E}) &::= \top \mid i \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \Diamond\phi \mid E\phi;\\
\mathcal{PH}(@,\downarrow) &::= \top \mid i \mid x \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \Diamond\phi \mid @_i\phi \mid @_x\phi \mid \downarrow x.\phi;\\
\mathcal{H} &::= \top \mid p \mid i \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \Diamond\phi;\\
\mathcal{H}(@) &::= \top \mid p \mid i \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \Diamond\phi \mid @_i\phi;\\
\mathcal{H}(\mathsf{E}) &::= \top \mid p \mid i \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \Diamond\phi \mid E\phi;\\
\mathcal{H}(@,\downarrow) &::= \top \mid p \mid i \mid x \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \Diamond\phi \mid @_i\phi \mid @_x\phi \mid \downarrow x.\phi;\\
\mathcal{HGL} &::= \top \mid p \mid i \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \Diamond\phi \mid \Diamond^{+}\phi \mid @_i\phi;\\
\mathcal{HGL}(\downarrow) &::= \top \mid p \mid i \mid x \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \Diamond\phi \mid \Diamond^{+}\phi \mid @_i\phi \mid @_x\phi \mid \downarrow x.\phi;\\
\mathcal{FHL} &::= \top \mid p \mid i \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \Diamond\phi \mid \Diamond^{-1}\phi \mid E\phi \mid @_i\phi \mid @_x\phi \mid \downarrow x.\phi.
\end{aligned}$$

The study of various modal languages in the context of graph property definability has proved to yield some valuable notions and examples, illustrating their comparative expressiveness. We will discuss the known results in this field, including the following.

1. No useful graph properties are definable in $\mathcal{ML}$ and $\mathcal{ML}^{\pm}$.

2. Full graphs and graphs without loops are finitely-definable in $\mathcal{PH}$ (and, consequently, in richer languages) [4].

3. Strongly connected graphs are finitely-definable in $\mathcal{H}(\mathsf{E})$.

4. Weak connectivity, k-regularity, clique number, independence number, diameter, radius, girth, minimum and maximum vertex degree are not definable in $\mathcal{H}(@)$

5. Strongly connected and acyclic graphs are finitely-definable in $\mathcal{HGL}$ [3].

6. Hamiltonian graphs are co-definable in $\mathcal{HGL}$ [3].

7. Every graph property in NP is co-definable in $\mathcal{FHL}$.

We will compare different extensions of the modal language from the point of view of graph problems, and provide proof sketches for the main results.

References

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