Presburger Arithmetic and Visser's Conjecture

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Abstract

Presburger Arithmetic the true theory of natural numbers with addition. We show that the interpretations of Presburger Arithmetic in itself are definably isomorphic to the trivial one, confirming the conjecture of A. Visser. To prove that, we develop a characterization of linear orderings interpretable in $(\mathbb{N}, +)$. We show that all interpretable linear orderings can be expressed as a restriction of the lexicographical ordering on $\mathbb{Z}^k$ for some k to some Presburger-definable set. This generalizes the results of [10] where the one-dimensional result was proven.

This talk is based on a joint work with Fedor Pakhomov.

Presburger Arithmetic **PrA** [7] is the true theory of natural numbers with addition. Unlike Peano Arithmetic **PA**, it is complete, decidable and admits quantifier elimination in an extension of its language.

A *reflexive* arithmetical theory ([9, p. 13]) is a theory that can prove the consistency of all its finitely axiomatizable subtheories. Peano Arithmetic **PA** and Zermelo-Fraenkel set theory **ZF** are among well-known reflexive theories. For sequential theories reflexivity implies that the theory cannot be interpreted in any of its finite subtheories. A. Visser has conjectured that this purely interpretational-theoretic property holds for **PrA** as well. Note that Presburger Arithmetic, unlike sequential theories, cannot encode tuples of natural numbers by single natural numbers, and thus, for interpretations in Presburger Arithmetic it is important whether individual objects are interpreted by individual objects (one-dimensional interpretations) or by tuples of objects of some fixed length m (m-dimensional interpretations).

As shown in [11], Visser's conjecture follows from the following statement:

Theorem 1. *Let $\mathfrak{A}$ be a model of* **PrA** *interpreted in* $(\mathbb{N}, +)$. *Then $\mathfrak{A}$ is isomorphic to* $(\mathbb{N}, +)$ *and, moreover, the isomorphism is definable in* $(\mathbb{N}, +)$.

In the work [10], we have established that Visser's conjecture holds for one-dimensional interpretations. We establish that by studying the interpretation of the ordering on $\mathfrak{A}$ induced by the interpretation.

We showed that each linear order that is interpretable in $(\mathbb{N}, +)$ is *scattered*, i.e. it doesn't contain a dense suborder. Moreover, we are able to give an estimation for Cantor-Bendixson ranks of the orders [3] (for a more precise estimation we use a slightly different notion of VD_*-rank from [5]):

Theorem 2 ([10], Theorem 4.3)**.** *All linear orderings m-dimensionally interpretable in* $(\mathbb{N}, +)$ *have the VD_*-rank at most m.*

Already for $n \geq 2$ the rank condition is far from sufficient. In order to produce a multi-dimensional version of Theorem 1, we establish a necessary and sufficient condition on the linear ordering interpretability. Turns out that the following holds:

Theorem 3. *A linear ordering $(L, \prec)$ is m-dimensionally interpretable in $(\mathbb{N}, +)$ for some $m \geq 1$ if and only if there exists some $k \in \mathbb{N}$ and a* **PrA***-definable set $\mathcal{D} \in \mathbb{Z}^k$ such that L is isomorphic to the restriction of the lexicographic ordering on $\mathbb{Z}^k$ to $\mathcal{D}$.*

From this description we derive the multi-dimensional Visser's conjecture.

Theorem 4 (Visser's Conjecture, Multi-Dimensional Version)**.** *For any model* $\mathfrak{A}$ *of* **PrA** *that is* m*-dimensionally interpreted in the model* $(\mathbb{N},+)$ *(*$m \geq 2$*),* $\mathfrak{A}$ *is* definably *isomorphic to* $(\mathbb{N},+)$*.*

The results are provided in the article [6].

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